Indian Statistical Institute
Mid-Semestral Examination
Algebra II - BMath I

Max Marks: 30 Time: 150 minutes.

Answer all questions. You may use Theorems stated/proved in the class after correctly stating them. You may use results not discussed in the class only after proving them.

(1) Decide whether the following statements are TRUE or FALSE. Answers that are not accompanied by a correct justification will not be awarded any marks.
   (a) There exists a 4×3 real matrix A and a 3×4 real matrix B such that the column vectors of AB are linearly independent.
   (b) If V is a vector space of dimension 8 and $T : V \longrightarrow V$ a linear transformation such that $T \circ T = 0$, then $\text{rank}(T) \leq 4$.
   (c) Suppose A and B are 2×2 real matrices such that $AX \neq BX$ for all non-zero X. Then $A - B$ is invertible.
   (d) Let $A = (a_{ij})$ be a square matrix with real entries such that $\sum_j a_{ij} = 1$ for all i. Then there exists a non zero X such that $AX = X$.
   (e) If V is a vector space over a field F and $W \neq \{0\}$ a subspace such that $V/W \cong V$ then V is infinite dimensional. [2 × 5 = 10]

(2) (a) Show that the system $AX = B$ where $B = (1, 2, 3)^t$ and

$$A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 7 & -5 & -1 \end{pmatrix}$$

   has no solutions. [5]
   (b) Prove that a square matrix A is invertible if and only if its columns are linearly independent. [5]

(3) (a) Consider the vector space $V = \mathbb{R}$ over the field $\mathbb{Q}$ of rational numbers. Exhibit explicitly a linearly independent set $L = (v_1, v_2, v_3)$ of vectors in V. [5]
   (b) Let V be the vector space of all polynomials with real coefficients of degree at most 2. Let t be a fixed real number and define $p_1(x) = 1$, $p_2(x) = x + t$ and $p_3(x) = (x + t)^2$. Prove that $B = (p_1, p_2, p_3)$ is a basis of V. If $p(x) = a_0 + a_1 x + a_2 x^2$, what are the coordinates of p in the ordered basis B? [5]